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Calculating the charged particle stopping power exactly to leading and next-to-leading order

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Abstract. I will discuss a new method for calculating transport quantities, such as the charged particle stopping power, in a weakly to moderately coupled plasma. This method, called dimensional continuation, lies within the framework of convergent kinetic equations, and it is powerful enough to allow for systematic perturbative expansions in the plasma coupling constant. In particular, it provides an *exact* evaluation of the stopping power to leading and next-to-leading order in the plasma coupling, with the systematic error being of cubic order. Consequently, the calculation is near-exact for a weakly coupled plasma, and quite accurate for a moderately coupled plasma. The leading order term in this expansion has been known since the classic work of Spitzer. In contrast, the next-to-leading order term has been calculated only recently by Brown, Preston, and Singleton (BPS), using the aforementioned method, to account for *all* short- and long-distance physics accurate to second order in the plasma coupling, including an exact treatment of the quantum-to-classical scattering transition. Under conditions relevant for inertial confinement fusion, BPS find the alpha particle range in the DT plasma to be about 30% longer than typical model predictions in the literature. Preliminary numerical studies suggest that this renders the ignition threshold proportionally higher, thereby having potential adverse implications for upcoming high energy density facilities. Since the key ideas behind the BPS calculation are possibly unfamiliar to plasma physicists, and the implications might be important, I will use this opportunity to explain the method in a pedagogical fashion.

For a weakly to moderately coupled plasma, the charged particle stopping power dE/dx was recently calculated from first principles in Ref. [1] using the method of dimensional continuation [2]. While the calculational techniques were imported from quantum field theory, the calculation itself lies squarely within the standard framework of convergent kinetic equations. I will assume some familiarity with Ref. [1], although a much shorter and self-contained exposition can be consulted in Ref. [3]. For ease of presentation, I will work here with a one component plasma of charge e, temperature T, and number density n, although the work described above is in the context of a general multi-component plasma. Our starting point will be the fact that any plasma quantity can be written as a series expansion in *integer* powers of a dimensionless plasma parameter [4]. For example, the stopping power takes the form

$$\frac{dE}{dx} = -\underbrace{A\,g^2\ln g}_{\text{LO}} + \underbrace{B\,g^2}_{\text{NLO}} + \underbrace{\mathcal{O}(g^3)}_{\text{error}} \;, \quad \text{with coupling parameter} \quad g \;=\; \frac{e^2\kappa_{\rm D}}{T} \;, \qquad (1)$$

where $\kappa_{\rm D}^2 = 4\pi\, e^2\, n/T$. I have indicated the leading order in g term (LO) and the next-to-leading order term (NLO) in Eq. (1), while the minus sign in the LO term is a convention that renders A positive when energy is transferred to the plasma. Note that nonanalytic terms such as $\ln g$ can also appear in the g-expansion. For the process of energy exchange via Coulomb interactions,

this non-analyticity arises from the competition between disparate physical length scales. The coupling g is just the ratio of the Coulomb energy, for two particles a Debye length $\lambda_{\rm D} = \kappa_{\rm D}^{-1}$ apart, to the thermal kinetic energy as measured by T. To get a feel for the numbers, one finds $g = 0.04$ at the center of the sun, while in the ICF ignition regime g can be smaller. In such cases, *provided* we can calculate A and B, expression (1) therefore gives an accurate evaluation of dE/dx.

The coefficient A was first calculated by Spitzer some time ago, while B was recently calculated in Ref. [1] using a regularization technique from quantum field theory called dimensional continuation. It is convenient to define the dimensionless coefficient C by $B = -A\,\ln C$, along with $K = A\,g^2$, and to then express the stopping power as

$$\frac{dE}{dx} \;=\; K\,\ln\Lambda_{\rm coul} \,+\, \mathcal{O}(g^3)\ ,\ \text{with}\quad \ln\Lambda_{\rm coul} = -\ln\{\mathrm{C\,g}\}\ . \tag{2}$$

We see, then, that knowing the next-to-leading order term is equivalent to knowing the exact coefficient C under the logarithm. In the extreme classical and quantum limits, the logarithm can be written as the ratio of two length scales

$$\ln\Lambda_{\rm coul} = \ln\!\left\{\frac{b_{\rm max}}{b_{\rm min}}\right\}\ , \tag{3}$$

where $b_{\rm max}$ is set by the Debye screening length $\kappa_{\rm D}^{-1}$, and the scale $b_{\rm min}$ is set either by the distance of closest approach in the extreme classical limit, or the thermal De Broglie length in the extreme quantum regime. In either case, we find $b_{\rm min}/b_{\rm max} \propto g$, and we see that the g-dependence in the Coulomb logarithm arises quite naturally. Reference [1] can therefore be thought of as a calculation of the Coulomb logarithm, including the exact interpolation between the extreme classical and quantum limits.

Let us now turn to convergent kinetic equations. As suggested by Ref. [5], one can view the Boltzmann and Leonard-Balescu equations as providing complementary physics since they both succeed and fail in complementary regimes. The Boltzmann equation (BE) gets the short-distance physics correct, while the Leonard-Balescu equation (LBE) captures the long-distance physics; conversely, the BE and the LBE miss the long- and short-distance physics, respectively. This complementarity motivates a class of kinetic equations of the form [6]

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f = B[f] + L[f] - R[f]\ , \tag{4}$$

where $R[f]$ is a carefully chosen "regulating kernel" designed to "subtract" the long-distance divergence from the scattering kernel $B[f]$ of the BE and the short-distance divergence from $L[f]$ of the LBE. Furthermore, the kernel $R[f]$ must also preserve the correct physics in the complementary regimes, namely, it must not damage the correct short-distance physics of the BE and the correct long-distance physics of the LBE. Reference [1] can be viewed as a systematic and rigorous implementation of this procedure, albeit in a more abstract form, accurate to second order in g (with more work, one could systematically calculate to third and higher order in g).

We now examine how dimensional continuation regulates the kinetic equations. For simplicity, we concentrate on the classical regime, although to second order in g, quantum mechanics can be included by using the quantum scattering amplitude in $B[f]$. The classical BBGKY hierarchy for the Coulomb potential is well defined and finite. We run into divergences only when truncating the hierarchy to derive lower-order kinetic equations, such as the Boltzmann and the Lenard-Balescu equations. Interestingly, this truncation problem occurs for the Coulomb potential, and only then in three spatial dimensions $\nu = 3$. Therefore, we can regulate the theory, rendering it completely finite and well defined, by performing the integrals in an arbitrary number of

dimensions ν. Upon using this procedure, logarithmic divergences in three dimensions become finite simple poles $1/(\nu-3)$ in arbitrary dimensions. One can then work entirely with finite quantities. In the case of the stopping power, we find that the long-distance pole from the BE exactly cancels the short-distance pole from the LBE, and the result is therefore finite when we set $\nu = 3$ at the end of the calculation. This provides a finite and well-defined result, obtained from a regularization prescription constructed in a consistent fashion at *all length and energy scales*. This is a common and time honored regularization procedure in quantum field theory, where it is called dimensional regularization.

I will now review some of the more salient features of the method. Let $\mathbf{x}_\nu$ and $\mathbf{v}_\nu$ denote the ν-dimensional position and velocity vectors of a particle. The Coulomb potential for two particles separated a distance $r = |\mathbf{x}_\nu - \mathbf{x}'_\nu|$ is $V_\nu(r) = C_\nu\, e^2/r^{\nu-2}$, where $C_\nu = \Gamma(\nu/2-1)/4\pi^{\nu/2}$ is a spatially dependent geometric factor.[1] The distribution function f_ν will be defined so that $f_\nu(\mathbf{x}_\nu, \mathbf{v}_\nu, t)\, d^\nu x\, d^\nu v$ gives the number of particles in a small hyper-volume $d^\nu x$ about $\mathbf{x}_\nu$ and $d^\nu v$ about $\mathbf{v}_\nu$ at time t. We can define multi-point correlation functions in a similar manner, and in this way we can construct the BBGKY hierarchy in an arbitrary number of dimensions. In dimensions $\nu > 3$, the standard textbook derivation of the BE goes through without an infrared divergent scattering kernel $B_\nu[f]$. Furthermore, since the ν-dimensional Coulomb potential $V_\nu(r) \propto 1/r^{\nu-2}$ emphasizes short-distance over long-distance physics when $\nu > 3$, the BBGKY hierarchy reduces to the Boltzmann equation to *leading* order in g in these dimensions:

$$\text{BBGKY} \Rightarrow \frac{\partial f_\nu}{\partial t} + \mathbf{v}_\nu \cdot \boldsymbol{\nabla}_x\, f_\nu = B_\nu[f] \quad \text{to LO in } g \text{ for } \nu > 3\ . \tag{5}$$

Here, the ν-dimensional spatial gradient has been denoted by $\boldsymbol{\nabla}_x$. Conversely, in dimensions $\nu < 3$, the Coulomb potential $V_\nu(r)$ emphasizes long-distance physics over short-distance effects, and consequently, to leading order in g, the BBGKY hierarchy reduces to the Lenard-Balescu equation in this spacial regime:

$$\text{BBGKY} \Rightarrow \frac{\partial f_\nu}{\partial t} + \mathbf{v}_\nu \cdot \boldsymbol{\nabla}_x\, f_\nu = L_\nu[f] \quad \text{to LO in } g \text{ for } \nu < 3\ , \tag{6}$$

where the scattering kernel in the LBE is $L_\nu[f]$. Space does not permit us to write down the exact forms of $B_\nu[f]$ and $L_\nu[f]$ here, but one may consult Ref. [1] for the expressions. These kinetic equations allow one to calculate the stopping power in $\nu > 3$ and $\nu < 3$, the results of which are presented in Sections 8 and 7 of Ref. [1], respectively. The calculation involves performing a series of momentum and wave number integrals in arbitrary dimensions ν, and reduces to the form

$$\frac{dE^>}{dx} \;=\; H(\nu)\,\frac{g^2}{\nu-3} + \mathcal{O}(\nu-3) \qquad :\ \text{LO in } g \text{ when } \nu > 3\ , \tag{7}$$

$$\frac{dE^<}{dx} \;=\; G(\nu)\,\frac{g^{\nu-1}}{3-\nu} + \mathcal{O}(3-\nu) \qquad :\ \text{LO in } g \text{ when } \nu < 3\ . \tag{8}$$

The analytic expressions for $H(\nu)$ and $G(\nu)$ are rather complicated,[2] and space does not permit their reproduction here. In this paper, we are only interested in their analytic properties as a function of ν. In particular, the coefficients $H(\nu)$ and $G(\nu)$ can be expanded in powers of $\epsilon = \nu - 3$, and we find

$$H(\nu) = -A + \epsilon\, H_1 + \mathcal{O}(\epsilon^2) \quad \text{and} \quad G(\nu) = -A + \epsilon\, G_1 + \mathcal{O}(\epsilon^2)\ . \tag{9}$$

[1] See Ref. [3] for more details.

[2] For the related process of electron-ion temperature equilibration, in contrast, the expressions for $H(\nu)$ and $G(\nu)$ are quite simple.

For our purposes, we do not require the exact forms of H_1, G_1, nor that of the leading term A. It is sufficient to note that the leading terms in Eq. (9) are equal, so that $H(\nu \equiv 3) = G(\nu \equiv 3)$. This is a fact that arises from the calculation itself, as it must, and it should be emphasized that this equality is not arbitrarily imposed by hand. It is a *crucial* point that the leading terms are identical, as this will allow the short- and long-distance poles to cancel, thereby giving a finite result.

Since the rates $dE^>/dx$ of Eq. (7) and $dE^</dx$ of Eq. (8) were calculated in mutually exclusive dimensional regimes, one might think that they cannot be compared. However, even though Eq. (8) was originally calculated in $\nu < 3$ for integer values of ν, we can analytically continue[3] the quantity $dE^</dx$ (viewed as a function of dimension ν) to real values of ν with $\nu > 3$. We can then directly compare Eqs. (7) and (8). Upon writing the g-dependence of Eq. (8) as $g^{2+(\nu-3)}$, when $\nu > 3$ we see that Eq. (8) is indeed higher order in g than Eq. (7):

$$\frac{dE^<}{dt} \;=\; -G(\nu)\,\frac{g^{2+(\nu-3)}}{\nu-3} + \mathcal{O}(\nu-3) \quad : \text{ NLO in } g \text{ when } \nu > 3 \;. \tag{10}$$

By power counting arguments, no powers of g between g^2 and $g^{\nu-1}$ can occur in Eq. (7) for $\nu > 3$, and therefore Eq. (8) indeed provides the correct next-to-leading order term in g when the dimension is analytically continued to $\nu > 3$, The individual pole-terms in Eqs. (7) and (10) will cancel giving a finite result when the leading and next-to-leading order terms are added. The resulting finite quantity will therefore be accurate to leading and next-to-leading order in g as the $\nu \to 3$ limit is taken:

$$\frac{dE}{dx} = \lim_{\nu\to 3^+}\left[\underbrace{\frac{dE^>}{dx}}_{\text{LO}} + \underbrace{\frac{dE^<}{dx}}_{\text{NLO}}\right] + \mathcal{O}(g^3) \;. \tag{11}$$

Note that this does not lead to any form of "double counting" since we are merely adding the next-to-leading order term (10) to the leading order term (7) at a common value of $\nu > 3$. We are now in a position to evaluate the limit in Eq. (11). Defining $\epsilon = \nu - 3$ as before, note that $g^\epsilon = \exp\{\epsilon \ln g\} = 1 + \epsilon \ln g + \mathcal{O}(\epsilon^2)$, which gives the relation

$$\frac{g^\epsilon}{\epsilon} = \frac{1}{\epsilon} + \ln g + \mathcal{O}(\epsilon) \;. \tag{12}$$

Substituting Eq. (12) into Eq. (10), adding this result to Eq. (7), and then taking the limit gives

$$\frac{dE}{dx} = -A\,g^2 \ln g + B\,g^2 + \mathcal{O}(g^3) \;, \tag{13}$$

with $B = H_1 - G_1$, in agreement with Eq. (1). In this way, BPS has calculated the charged particle stopping power accurate to leading order and next-to-leading order in g.

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[3] In the same way that the factorial function $n!$ on the positive integers can be generalized to the Gamma function $\Gamma(z)$ over the complex plane, including both the positive and negative real axes.